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On the Reciprocity of State Vectors in Boundary Value Models*

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Abstract

In this report, we prove that the state vector of a two-point boundary value model is a reciprocal process.

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1. Introduction

When modeling physical systems in which one or more of the independent variables is spatial, the *a priori* information is usually given in the form of boundary conditions rather than initial conditions. Such systems arise in acoustics and oceanography, for example. Consider the following linear, two-point boundary value model for $t_0 \leq t \leq t_1$:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$r = V_0 x(t_0) + V_1 x(t_1) \tag{2}$$

where u is a Gaussian, zero-mean, unit intensity white noise with m components, r is a Gaussian, zero-mean random vector with n components and is independent of u, x is the n-component state vector, and A, B, V_0, V_1 are appropriate matrices.

Structural properties of such two-point boundary value models have been studied in [5]-[6]. Estimation problems based on such models have been treated in [2]-[3], [7]-[8]. As is customary, we will assume that the model is *well-posed*; i.e., that u and r give rise to a unique x. This will be the case if and only if the matrix $V_0 + V_1\Phi(t_1,t_0)$ is nonsingular, in which case we can assume, without loss of generality, that

$$V_0 + V_1\Phi(t_1,t_0) = I$$

where Φ is the state transition matrix of (1).

Since, in general, $x(t_0)$ is correlated with u, the state vector x is not Markovian. However, as we shall prove, it does possess the weaker property of reciprocity, which will be defined presently. First, we note that Krener

stated this result in [5] and sketched a method of proof for it. However, we have been unable to complete the proof along the suggested lines, and were thus motivated to find a different approach.

A process is *reciprocal* on an interval if, given any subinterval, the values of the process inside the subinterval are conditionally independent of the values outside the subinterval, given the values of the process at the two endpoints of the subinterval. For information on reciprocal processes, see [1], [4]-[5] and the references therein. In a recent paper [4], Carmichael *et al* give a very useful characterization of reciprocal processes, based on the projection theorem. They prove that a process x is reciprocal if and only if, for all $a \in [t_0\ ,\ t_1]$, the two (one-sided) error processes e_1 and e_2 given by

$$\begin{aligned} e_1(t) &= x(t) - E[x(t)|x(a)] \qquad \text{for } t_0 \le t \le a \\ e_2(t) &= x(t) - E[x(t)|x(a)] \qquad \text{for } a \le t \le t_1 \end{aligned}$$

are Markovian. Our proof is based on this result, and on the fact that if a Gaussian process has a semi-separable covariance function, it is Markovian.

2. The Main Result

The boundary value model (1), (2) can be solved for $x(a)$, for all $a \in [t_0\ ,\ t_1]$, as follows:

$$x(a) = \Phi(a,t_0)r + \int_{t_0}^{t_1} G(a,\tau)Bu(\tau)d\tau$$

where G is the Green's function given by

$$G(a,t) = \begin{cases} -\Phi(a,t_0)V_1\Phi(t_1,t) & \text{for } a < t \\ \Phi(a,t_0)V_0\Phi(t_0,t) & \text{for } a > t \end{cases} \tag{3}$$

Let $R(a,a)$ denote the covariance matrix of $x(a)$. We assume this matrix is nonsingular for all $a \in [t_0 \ , \ t_1]$. From (1) and the definitions of e_1 and e_2 given above, it is clear that for $t_0 \le t \le a$

$$\dot{e}_1(t) = Ae_1(t) + B\tilde{u}(t), \quad e_1(a) = 0 \tag{4}$$

and for $a \le t \le t_1$

$$\dot{e}_2(t) = Ae_2(t) + B\tilde{u}(t), \quad e_2(a) = 0 \tag{5}$$

where

$$\begin{aligned}\tilde{u}(t) &= u(t) - E[u(t)|x(a)] \\ &= u(t) - E[u(t)x'(a)]R^{-1}(a,a)x(a) \\ &= u(t) - B'G'(a,t)R^{-1}(a,a)x(a)\end{aligned}$$

and

$$E[\tilde{u}(t)\tilde{u}'(s)] = I\delta(t-s) - B'G'(a,t)R^{-1}(a,a)G(a,s)B \tag{6}$$

Theorem: The state vector process x is reciprocal.

Proof: As discussed above, it is sufficient to prove that e_1 and e_2 are Markovian for all $a \in [t_0 \ , \ t_1]$. Consider first the case of e_2. For $a \le t \le t_1$ and $a \le s \le t_1$, its covariance function is given by

$$\begin{aligned}K_2(t,s) &= E[e_2(t)e_2'(s)] \\ &= \int_a^t \int_a^s \Phi(t,\tau)B\{E[\tilde{u}(\tau)\tilde{u}'(\sigma)]\}B'\Phi'(s,\sigma)d\sigma d\tau\end{aligned}$$

where we have used (5). Substituting in the above using (6) produces

$$\begin{aligned}K_2(t,s) = &\int_a^{\min(t,s)} \Phi(t,\tau)BB'\Phi'(s,\tau)d\tau \\ &-\left[\int_a^t \Phi(t,\tau)BB'G'(a,\tau)d\tau\right]R^{-1}(a,a)\left[\int_a^s G(a,\tau)BB'\Phi'(s,\tau)d\tau\right]\end{aligned}$$

and using equation (3) for the Green's function, we get

$$K_2(t,s) = \int_a^{\min(t,s)} \Phi(t,\tau)BB'\Phi'(s,\tau)d\tau$$
$$-\left[\int_a^t \Phi(t,\tau)BB'\Phi'(t_1,\tau)d\tau\right]V_1'\Phi'(a,t_0)R^{-1}(a,a)\Phi(a,t_0)V_1\left[\int_a^s \Phi(t_1,\tau)BB'\Phi'(s,\tau)d\tau\right]$$
$$= \begin{cases} C_2(t)D_2(s) & \text{for } a \le s \le t \le t_1 \\ D_2'(t)C_2'(s) & \text{for } a \le t \le s \le t_1 \end{cases}$$

where

$$C_2(t) = \Phi(t,t_1) - \left[\int_a^t \Phi(t,\tau)BB'\Phi'(t_1,\tau)d\tau\right]V_1'\Phi'(a,t_0)R^{-1}(a,a)\Phi(a,t_0)V_1$$
$$D_2(t) = \int_a^t \Phi(t_1,\tau)BB'\Phi'(t,\tau)d\tau$$

Since its covariance function is semi-separable, $e_2(t)$, $a \le t \le t_1$, is Markovian for all $a \in [t_0\ ,\ t_1]$. Using the same approach, it can be shown that, for $t_0 \le t \le a$ and $t_0 \le s \le a$,

$$K_1(t,s) = E[e_1(t)e_1'(s)]$$
$$= \int_{\max(t,s)}^a \Phi(t,\tau)BB'\Phi'(s,\tau)d\tau$$
$$-\left[\int_t^a \Phi(t,\tau)BB'\Phi'(t_0,\tau)d\tau\right]V_0'\Phi'(a,t_0)R^{-1}(a,a)\Phi(a,t_0)V_0\left[\int_s^a \Phi(t_0,\tau)BB'\Phi'(s,\tau)d\tau\right]$$
$$= \begin{cases} C_1(t)D_1(s) & \text{for } t_0 \le t \le s \le a \\ D_1'(t)C_1'(s) & \text{for } t_0 \le s \le t \le a \end{cases}$$

where

$$C_1(t) = \Phi(t,t_0) - \left[\int_t^a \Phi(t,\tau)BB'\Phi'(t_0,\tau)d\tau\right]V_0'\Phi'(a,t_0)R^{-1}(a,a)\Phi(a,t_0)V_0$$
$$D_1(t) = \int_t^a \Phi(t_0,\tau)BB'\Phi'(t,\tau)d\tau$$

Therefore, $e_1(t)$, $t_0 \le t \le a$, is Markovian for all $a \in [t_0\ ,\ t_1]$, and the theorem is proved.

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